A Comparative Study of Mean Value Analysis and Convolution Algorithm for Queueing Networks

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***Abstract*- Present paper deals, comparative study between mean value analysis and convolution algorithm for queueing network models by exponentially distributed service time for single and multi-class system. The throughput, utilization, average response time and mean number of jobs were obtained using performance measures. Numerical illustrations have been carried out to examine the effect of various parameters on performance indices.**

Keywords- MVA, CA, Probability, Normal Constant, Utilization, Through-put & Response time.

I. INTRODUCTION

A queueing network model is a collection of service centers representing the system resources that provide service to a collection of customers that represent the users. The customer competition for the resource service corresponds to queueing into the service centers. The analysis of the queueing network models consists of evaluating a set of performance measures. Many researchers have studied these issues in different framework. Bruell and Balbo [1] analyzed the computational algorithm for closed queueing networks with finite servers. Buzen [2] obtained the convolution algorithm for product form queueing network. Hiseh and Lan [5] studied the two classes of performance bounds for closed queueing networks. Whitt [6] analyzed various performance measures for queueing networks.

Akyildiz [9] involving an approximation method for the throughput of finite closed queueing networks. Gordon and Newell [10] introduced queueing model for closed networks with exponential servers. The queueing discipline at all stations was assumed to be First Come First Served (FCFS). These results were extended to open, closed and mixed networks with several customer classes, non-exponentially distributed service times and different queueing disciplines. Bruno and Dallery [13] obtained the approximate solution of general closed queueing network with several classes of customer. Reiser and Lavenberg [14] derived MVA algorithm for closed multi chain queueing networks. Akyildiz and Bolch [15] presented an approximation of MVA for queueing networks containing multiple server stations. Balbo and Serazzi [16] established asymptotic for multi-class closed queueing networks with multiple bottlenecks. Liithi and Haring [17] considered MVA algorithm for separable closed queueing networks with one customer class consisting of load independent queueing centers as well as delay devices. Wang and Sevick [18] established experimental approach of MVA algorithm. Fillipowicz and Kwiecien [19] described queueing models performance analysis of different systems such as customer, communication, transportation networks and manufacturing.

The paper is organized some sections. In section 2 presents some preliminaries. Mean valve analysis and convolution algorithm are described for single class queueing networks and multi-class queueing networks with their performance measures along with their numerical illustration in section 3, 4 and 5 respectively. Section 6 concludes this paper.

II. PRELIMINARIES

- N = Number of users
- M = Number of devices
- X = System throughput

- X_i = Throughput rate (jobs per unit time)
- λ = Arrival rate
- α = Scaling factor
- μ = Service rate at queue i
- e_i = visit ratio of i^{th} user (i =1,2,. ,K)
- Z = Think time
- E = Maximum allowable error in queue length
- R = Response of the i^{th} device.
- Q_i = Average number of jobs at the i^{th} device
- D_i = Total demand of single device for queue i
- V_i = Average number of visit to the device i
- S_i = Service time per visit to the i^{th} device
- U_i = Utilization of the i^{th} device
- R_i = Response of the i^{th} device.
- P(n) = probability of queue length vector being.

III. Classification of Queueing Networks

A. *Single class queueing network*

In this section, we give the results for single class queueing networks using mean value analysis and convolution algorithm.

B. *Multi- class Queueing Networks*

We discuss, mean value analysis of closed queueing networks with product-form solution. The advantage of this method is to compute the performance measures. This method is based on two fundamental equations and it allow us to compute the mean value of measures of interest such as the mean response time, throughput and the mean number of jobs at each node. For the case of multi-server nodes (m_i> 1), it is necessary, to compute the marginal probabilities.

At the moment a job arrive at a node, it is certain that this job itself is not already in the queue of this node. Thus, there are only r-1 other jobs that could possibly interfere with the new arrival. The number of these at the node is simply the number there when only those (r-1) jobs are in the network.

IV. Performance Measures

Single –Class	
Mean Value Analysis	**Convolution Algorithm**
Mean value analysis is applied to close queueing networks with a variety of service discipline and service time distribution. We assumed the equations for all devices to be fixed capacity service center. If a device is a delay center, there is no waiting before service. **Response Times:** The response time at individual devices, for the single closed queueing network the response time using the general formula $R(N) = \sum_{i=1}^{M} V_i R_i(N)$ **System Throughput:** The system throughput using interactive response time formula $X(N) = \frac{N}{R(N)+Z}$ [where Z is think time] **Throughput:** The device throughput measured in term of job per second $X_i(N) = X(N)* V_i(N)$	Once the G(1), G(2), ……., G(N) are known , other performance measures can be computed as following: **Queue Length:** the probability of having j or more jobs at the i^{th} device. $Q_I = \sum_{J=1}^{n} P(n_i \geq j) = \sum_{j=1}^{N} y_i^j \frac{G(N-1)}{G(N)}$ Where $P(n_i \geq j) = y_i^j \frac{G(N-j)}{G(N)}$ **Utilization:** the device utilization under state probability for more than jobs at the device $U_i = y_i \frac{G(N-1)}{G(N)}$ **Throughputs:** the device throughputs are given by the utilization formula $X_i = \frac{U_i}{S_i}$ [Where S_i is the service time per visit to the i^{th}

Queue Length: The device queue length with N jobs in this system $Q_i(N) = X_i(N)\, R_i(N)$ $Q_i(N) = X(N) * V_i(N) * R_i(N)$ [using Throughput] **Utilization:** the utilization of the i^{th} device $U_i = X(N)\, S_i(N)\, V_i(N)$	device] **The system throughput** $X = \alpha \frac{G(N-1)}{G(N)}$ where α is scaling factor **Response Times:** the response time of a device is given by formula $R_i = \frac{Q_i}{XV_i}$ **[**WhereQ_i is average number of jobs at the i^{th} device & V_i is average number of visit tothe device i]
Multi-class	
Mean Value Analysis	**Convolution Algorithm**
Utilization: The utilization of the r-jobs with respect to N-nodes $\rho_{ir} = \frac{\lambda_{ir}}{m_i \mu_{ir}}.$ **Throughput:** The throughput measure of class r- jobs with N-nodes as per second $\lambda_r(k) = \frac{k_r}{\sum_{i=1}^{N} e_{ir} T_{ir}(k)}$ **Mean Number of Jobs:**We compute the mean number of r-jobs with N- nodes in this system $K_{ir}(k) = \lambda_r(k).T_{ir}(k).e_{ir}$ **Response Times:** The response time at individual states, for the multi-class closed queueing network the response time using the general formul $T_{ir}(k) \begin{cases} \frac{1}{\mu_{ir}}\left[1 + \sum_{s=1}^{R} K_{is}(k-1_r)\right] & \text{Case 1, 2 } (m_i = 1), \\ \frac{1}{\mu_{ir} m_i}\left[1 + \sum_{s=1}^{R} K_{ir}(k-1_r) + \sum_{j=0}^{m_{i-2}} (m_i - j - 1)\pi_i(j \mid k-1_r)\right] & \text{Case 1 } (m_i > 1), \\ \frac{1}{\mu_{ir}}, & \text{Case 3} \end{cases}$	**Utilization**(ρ_{ir})**:**The utilization(ρ_{ir})n of the ith node with respect to jobs of the rth class and if the service rates are constant, then we have the following simplification because of $\lambda_i/(m_i \mu_{ir})$: $\rho_{ir} = \frac{e_i}{m_i \mu_i}.\frac{G\,(K-1)}{G(K)}$ **Throughput (λ_{ir}):**the throughput$\lambda_{ir}$of class –r jobs at the ith node can be expressed as follows $\lambda_{ir} = e_{ir}\frac{G(K-1)}{G(K)}$ **Mean no. of Jobs (K_{ir}):** $K_{ir} = \sum k_r . p_i(k)$ **Mean Response Time (T_{ir}):** $T_{ir} = \frac{K_{ir}}{\lambda_{ir}}$

V. Numerical Illustrations

A. *Case I: Single class*

We consider single classqueueing network model with following inputs. Observation interval = 3600 seconds, no. of client requests = 7200, CPU busy time = 1020 seconds, Disk A busy time = 864 seconds and Disk B busy time = 1380 seconds, No. of visit (I/O requests for disk A = 44,200 and disk B = 50,400) with service rates $S_{cpu} = 0.1$, $S_A = 0.3$ sec. , $S_B = 0.2$ sec., $V_A = 6$ visit /client request to disk A & $V_B = 7$ visit /client request to disk B and $V_{CPU} = 14$ and total demand for $D_{cpu} = 0.14$, $D_A = 0.18$ & $D_B = 0.14$, Z= 3 sec. for closed queueing network, which is described in fig. 1 each user request makes ten I/O requests to disk A and disk B and also Consider 10 states of the system. These states and their state using Gordon and Newell's method probabilities are shown in table 2. For each state, we first determine the product $\prod_i^k y_i^{n_i}$, add all the product to find G(N), and divide each individual product by G(N) to get the desired probability. In table 3, we mention normalizing constant and table 4, we again evaluate the response time, system throughput and queue length using CA and also consider the scaling factor as α = 1/0.14 and $y_{CPU} = 1$, $y_A = 3$ and $y_B = 2$.

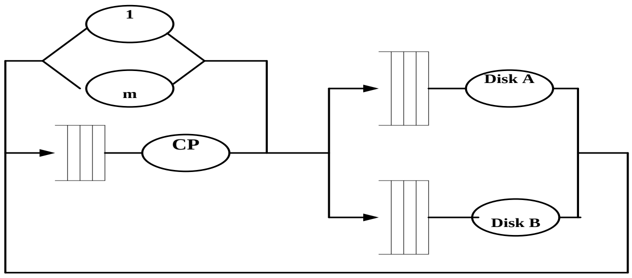

Figure 1: Single class queueing network

Table 1. Response time Vs Queue length for MVA

No. of iteration	**Response Time**				**System Throughput**	**Queue Length**		
	CPU	Disk A	Disk B	System		CPU	Disk A	Disk B
1	0.125	0.4	0.3	7.5	0.095	0.19	0.38	0.143
2	0.149	0.552	0.343	9.615	0.159	0.377	0.875	0.272
3	0.172	0.751	0.382	12.162	0.198	0.545	1.484	0.378
4	0.193	0.994	0.413	15.094	0.221	0.684	2.197	0.457
5	0.211	1.279	0.437	18.339	0.234	0.789	2.996	0.512
6	0.224	1.598	0.454	21.831	0.242	0.864	3.862	0.548
7	0.233	1.945	0.464	25.497	0.246	0.916	4.776	0.571
8	0.239	2.311	0.471	29.293	0.248	0.949	5.723	0.584
9	0.244	2.689	0.475	33.167	0.249	0.971	6.691	0.591
10	0.246	3.076	0.477	37.091	0.249	0.983	7.672	0.595

Table 2. Probability of system states

Number of Jobs at			**Numerator**	**Probability**
CPU	**Disk A**	**Disk B**	$\prod_i^k y_i^{n_i}$	
0	0	3	8	0.089
0	1	2	12	0.133
0	2	1	18	0.200
0	3	0	27	0.300
1	0	2	4	0.044
1	1	1	6	0.067
1	2	0	9	0.100
2	0	1	2	0.022
2	1	0	3	0.033
3	0	0	1	0.011
			G(N) = 90	1.000

Table 3. Normalizing Constants

N	$y_{CPU=1}$	$y_A = 3$	$y_B = 2$
0	1	1	1
1	1	4	6
2	1	13	25
3	1	40	90
4	1	121	301
5	1	364	966
6	1	11093	3025
7	1	3280	9330
8	1	9841	28501
9	1	29524	86526
10	1	88573	261625

The service times per visit to disk A and B are 300 and 200 milliseconds, respectively. Each request take 2 seconds for CPU time and the user think time is 3 seconds. In table 1, we calculate the response time, system throughput and queue length by using MVA.

Table 4. Response time Vs Queue length for Convolution

No. of Iteration	Response Time				System Throughput	Queue Length		
	CPU	Disk A	Disk B	System		CPU	Disk A	Disk B
1	0.010	0.070	0.040	0.84	16.702	0.167	0.501	0.334
2	0.012	0.105	0.053	1.16	23.996	0.281	1.081	0.641
3	0.013	0.144	0.065	1.50	27.776	0.355	1.712	0.904
4	0.014	0.168	0.076	1.73	29.820	0.405	2.154	1.138
5	0.014	0.240	0.087	2.25	31.192	0.438	3.204	1.364
6	0.014	0.299	0.093	2.65	31.892	0.458	3.993	1.481
7	0.015	0.358	0.101	3.19	32.382	0.474	4.967	1.631
8	0.015	0.415	0.107	3.45	32.690	0.496	5.816	1.752
9	0.015	0.463	0.107	3.73	32.984	0.486	6.551	1.763
10	0.016	0.523	0.111	4.17	33.082	0.492	7.423	1.832

B. Case II: Multi -class

We consider for the multi class queueing network (figure 4) with N = 4 nodes and K = 2 jobs classes. In class 1 the number of jobs is K_1 = 1 and in class 2, K_2 =2. According to figure 4, it is assumed that class switching is not allowed. The first node is of case 2, the second and third nodes are of case 1, and the fourth node is of case 3 with mean service times as $1/\mu_{11}$ = 1 sec, $1/\mu_{12}$ = 2 sec, $1/\mu_{21}$= 4 sec, $1/\mu_{22}$ = 5 sec, $1/\mu_{31}$ = 8 sec and $1/\mu_{32}$ = 10sec, $1/\mu_{41}$ = 12 sec,$1/\mu_{42}$ = 16 sec. we calculate the response time, system throughput and queue length by using MVA for multi-class as given in table 5. And also we consider same data for convolution techniques and we assume visit ratios are given as e_{11} = 1, e_{12} = 1, e_{21} = 0.4, e_{22} = 0.4, e_{31} = 0.4., e_{32} = 0.3, e_{41} = 0.2, e_{42} = 0.3

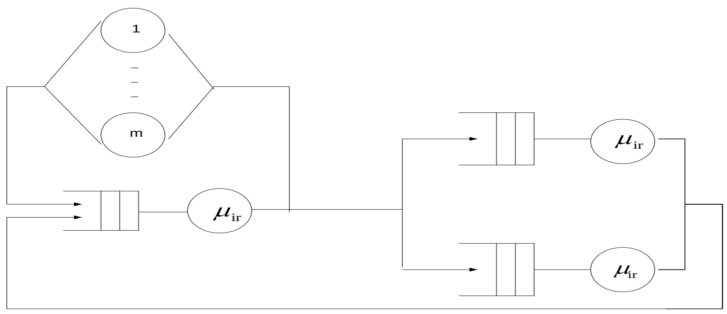

Figure 2: Multi class queueing network

First, we compute the function values $F_i(S_i)$, i =1, 2, 3, 4 and using figure 5 as given the states values in table 6. Now, we find out the normalization constant we compute the $G_n(k^{(n)})$with functions values $F_i(S_i)$ from the Equation. where $G_1(.) = F_1(.)$. for n = 2 we get $G_2(0,0) = 1$,$G_2(1,0) = 2.6$, $G_2(0,1) = 4$, $G_2(1,1) = 15.6$, $G_2(0,2) = 12$ and $G_2(1,2) = 62.4$.In similar, we can compute the values for $G_3(k^{(3)})$ and $G_4(k^{(4)})$ as summarized in given table 7.

Table 5. MVA for Multi Class Nodes

Nodes	1	2	3	4
Utilization (ρ_i)	0.122	0.941	0.084	0.232
Throughput (λ_i)	0.122	0.235	0.084	0.093
Mean Number of Jobs (K_{ir})	0.251	0.501	0.258	0.324
Mean response Time (T_{ir})	6.25	6.45	8.25	8.44

Table 6. State Values

State (S)	$F_1(S_1)$	$F_2(S_2)$	$F_3(S_3)$	$F_4(S_4)$
(0,0)	1	1	1	1
(1,0)	1	1.6	3.2	2.4
(0,1)	2	2	3	4.8
(1,1)	4	6.4	19.2	11.52
(0,2)	4	4	9	11.52
(1,2)	12	19.2	86.4	27.648

Table 7. Computed G_2 Values

k^n, $1 \leq n \leq N$	$G_2(k^{(2)})$	$G_3(k^{(3)})$	$G_4(k^{(4)})$
(0,0)	1	1	1
(1,0)	2.6	5.8	8.2
(0,1)	4	7	11.8
(1,1)	15.6	55.4	111.56
(0,2)	12	33	78.12
(1,2)	62.4	334.2	**854.424**

Table 8. Marginal Probabilities

$p_4(1,1)$	0.0324	$G_N^{(4)}(0,0)$	1
$p_4(1,1)$	0.0944	$G_N^{(4)}(0,1)$	7
$p_4(0,1)$	0.3112	$G_N^{(4)}(1,1)$	55.4
$p_4(1,0)$	0.0927	$G_N^{(4)}(0,2)$	33
$p_4(0,0)$	0.3911	$G_N^{(4)}(1,2)$	334.2
$p_3(1,2)$	0.1011	$G_N^{(3)}(0,0)$	1
$p_3(1,0)$	0.1600	$G_N^{(3)}(0,2)$	42.72
$p_3(1,1)$	0.1977	$G_N^{(3)}(0,1)$	8.8

Table 9. CA for Multi Class Nodes

Nodes	1	2	3	4
Utilization (ρ_{ir})	0.293	0.392	0.22	0.627
Throughput (λ_{ir})	0.037	0.039	0.018	0.039
Mean Number of Jobs (K_{ir})	0.459	0.678	0.219	0.627
Mean response Time (T_{ir})	12.54	17.31	11.97	15.99

Thus, the normalization constant G(K) = **854.424.** Now, we compute the marginal probabilities of 0 jobs of class 1 and 2 jobs of class 2 at node 4 with the help of the formula

$$p(S_1, S_2.., S_N) = \frac{1}{G}[p(S_1).p(S_2) \dots p(S_N)] \quad (1)$$

$$p_4(0,2) = \frac{F_4(0,2)}{G(K)} G_N^{(4)}(1,0) = 0.0782. \quad (2)$$

Where $G_N^{(4)}(1,0)$ can be obtained from the equation. (refer appendix).Similarly, other **marginal probabilities** at the nodes can be calculated in table 8 and we again evaluate the response time, system throughput and queue length by using CA for multi-class as given in table 9.

VI. Comparative Results for Single Class and Multi – Class Queueing Network

For single class, we have plotted the graph between response time is increase with the number of users and throughput. In figure 3, establishes that the response time increases with number of users. The MVA and it is appearing line when computed by CA is shown in figure 4. Here also throughput is increasing with number of users by CA but after large number of users, it becomes steady.

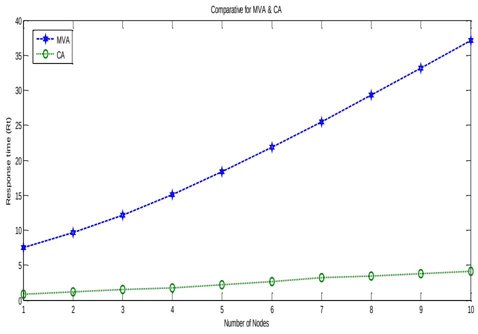

Figure 3: Response time

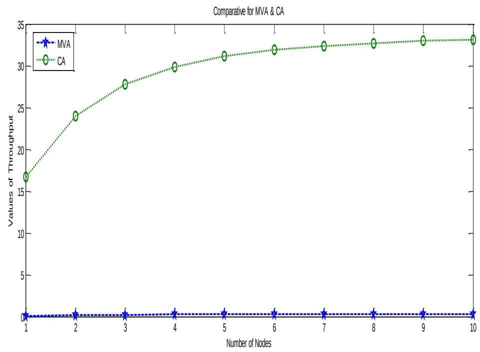

Figure 4: Throughput

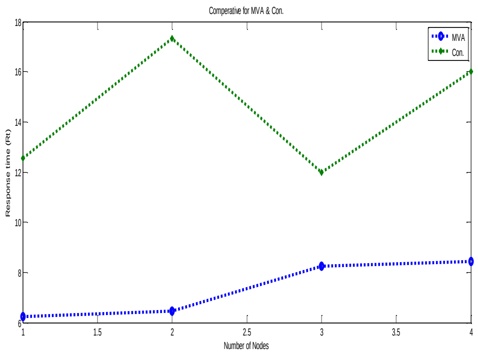

Figure 5: Response time

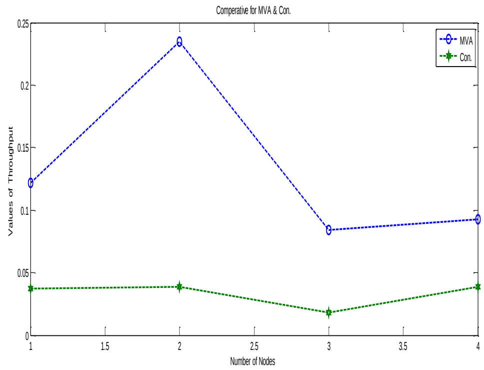

Figure 6: Throughput

A comparative steady of multi-class queueing network model using CA and MVA is presented in the figures 5 & 6. Figure 5 plots the graph between response time and number of nodes, when we apply MVA technique the graph is almost a straight line and by CA, it appear as a zigzag manner. Similarly when we draw the graph between throughput against number of nodes. Then it appears in opposite fashion as in figure 6.

VII. Conclusion

Queueing network models have been extensively applied to represent and analyze resource sharing system such as communication and computer systems and they have proved to be powerful and versatile tool for system performance evaluation and prediction. We describe the implementation and comparison of mean value technique and convolution algorithm for single and multi-class queueing networks. We have provided various performance indices at operation level. Numerical illustrations have been obtained for single and multi-class system

Appendix

Algorithms	
Single -class	
MVA	**Convolution**

Input:
R = System response time
Initialization: For i = 1 to K do $Q_i = 0$
Iterations:
for n = 1 to N do
begin
for i = 1 to K do
$R_i = \begin{cases} S_i(1+Q_i) & Fixed\ Capacity \\ S_i & Delay\ Centers \end{cases}$
$R(N) = \sum_{i=1}^{M} V_i R_i(N)$
$X(N) = \frac{N}{R(N)+Z}$
	Input:
	R = System response time
	U_i = utilization of the ith device
	P(n) = probability of queue length vector being n
	Scaling: $y_0 = \alpha Z$
	For i =1 to M do $y_i = \alpha S_i V_i$
	Initialization:
	G(0) = 1
	For n = 1 to N do G(n) = $G(n) = y_0^n / n!$
	Compute G(N):
	For i =1 to M do
	Begin

For i = 1 to K do
$Q_i(N) = X(N)^* V_i(N)^* R_i(N)$
End

For n= 1 to N do $G(n) = G(n) + y_i G(n - 1)$
End
Compute performance:

$$X = \alpha \frac{G(N-1)}{G(N)} \qquad U_i = X S_i V_i$$

$$Q_I = \sum_{J=1}^{N} y_i^j \frac{G(N-1)}{G(N)} \qquad R_i = \frac{Q_i}{X V_i}$$

$$R = \sum_{i=1}^{M} R_i V_i$$

Check: N = X(R+Z)

$$P(n_0, n_1, \ldots\ldots, n_M) = \frac{y_0^{n_0}\, y_1^{n_1}\, y_2^{n_2} \ldots\ldots y_M^{n_M}}{n_0! G(N)}$$

$$P(n_i \geq j) = y_i^j \frac{G(N-j)}{G(N)}, \quad i = 1, 2, 3, \ldots\ldots\ldots, M$$

$$P(n_i = j) = \frac{y_i^j}{G(N)}\left[G(N-1) - y_i G(N-j-1)\right], \; i = 1, 2, 3, \ldots., M$$

$$P(n_i \geq j, n_k \geq l) = y_i^j y_k^l \frac{G(N-j-1)}{G(N)}, \quad i = 1, 2, 3, \ldots\ldots\ldots, M$$

Multi-class

MVA

Step 1: Initialization. For i =1… N, r = 1 R, j =1 ., (m_i - 1):
$K_{ir}(0, 0, \ldots\ldots, 0) = 0, \quad \pi_i(0|0) = 1, \; \pi_i(j|0) = 0.$

Step 2: Iteration: k =0… K:
Step 2.1: For i = 1, ….N and r =1, ….., R, compute the mean response time of class-r jobs at the i^{th} node:

$$T_{ir}(k) = \begin{cases} \frac{1}{\mu_{ir}}\left[1 + \sum_{s=1}^{R} K_{is}(k - 1_r)\right] & Case\ 1,2\ (m_i = 1), \\ \frac{1}{\mu_{ir} m_i}\left[1 + \sum_{s=1}^{R} K_{ir}(k - 1_r) + \sum_{j=0}^{m_{i-2}} (m_i - j - 1)\pi_i(j|k - 1_r)\right] & Case\ 1\ (m_i > 1), \\ \frac{1}{\mu_{ir}}, & Case\ 3 \end{cases}$$

Here $(k - 1_r) = (k_1, k_2, \ldots\ldots k_r - 1, \ldots\ldots\ldots, k_R)$ is the population vector with one class-r job less in the system.

Case 1: M/M/m – FCFS. The service rates for different job classes must be equal. For example of case 1 nodes are input/output (I/O) devices or disks.

Case 2: M/G/1 – PS. The CPU of a computer system can very often be modeled as a case 2 node.

Case 3: M/G/∞ - (infinite server). Terminal can be modeled as case 3 nodes.
The probability that there are j jobs at the i^{th} node (j = 1,……., (m_i - 1) given that the network is in state k is given by

$$\pi_i(j|k) = \frac{1}{j}\left[\sum_{r=1}^{R} \frac{e_{ir}}{\mu_{ir}} \lambda_r(k) \pi_i(j-1|k-1_r)\right]$$

and for j=0 by

$$\pi_i(0|k) = 1 - \frac{1}{m_i}\left[\sum_{r=1}^{R} \frac{e_{ir}}{\mu_{ir}} \lambda_r(k) + \sum_{j=1}^{m_i - 1} (m_i - j)\, \pi_i(j|k)\right],$$

Where e_{ir} can be computed by

$$e_{ir} = \sum_{s \in C_q} e_{js} p_{js,ir} \qquad for\ j = 1, \ldots, N$$

Step 2.2: For r =1, ……, R, compute the throughput:

$$\lambda_r(k) = \frac{k_r}{\sum_{i=1}^{N} e_{ir} T_{ir}(k)}$$

Convolution

The Convolution method can be extended to the case of multi class closed product–form queueing networks. According to the BCMP theorem with state probabilities for closed product form queueing networks are given by

$p(S_1, S_2 \ldots, S_N) = \frac{1}{G(K)} \prod_{i=1}^{N} F_i(S_i)$

Where $F_i(S_i)$is the function, the determination of normalization constant of the multi class closed queueing network

$G(K) = \sum_{\sum_{i=1}^{N} S_i = K} \prod_{i=1}^{N} F_i(S_i)$

is analogous to the single class case. For now, we assume that class switching is not allowed and thus the number of jobs in each class is constant, we define for k = 0…….. K the auxiliary functions

$G_n(K) = \sum_{\sum_{i=1}^{N} S_i = K} \prod_{i=1}^{n} F_i(S_i)$ (3)

With the initial condition $G_1(.) = F_1(.)$. for n > 1 we also have

$G_n(k) = \sum_{j_1=0}^{k_1} - - \sum_{j_R=0}^{k_R} F_n(j). G_{n-1}(k - j).$ (4)

The convolution of the normalization constant was simplified by [12] and [20] by introducing a much simple iteration formula for $G(K) = G_N(K)$for this purpose, we define the vector

$k^{(n)} = \sum_{i=1}^{n} S_i = \left(k_1^{(n)}, \ldots\ldots\ldots, k_R^{(n)}\right).$ (5)

Where $k_r^{(n)}, 1 \leq r \leq R$ is the overall number of jobs of class r at the nodes 1,2,….., n-1, n. we have

$k_r^{(n)} = \sum_{i=1}^{n} k_{ir}$ and $k^{(N)} = K.$

With the help of these expressions, it is possible to rewrite equation (4) in the following manner:

$G_n\left(k^{(n)}\right) = \sum_{\sum_{i=1}^{n} S_i = k^{(n)}} \prod_{i=1}^{n} F_i(S_i)$ (6)

Thus

$$G_n\left(k^{(n)}\right) = \sum_{k^{(n-1)} + S_n = k^{(n)}} \sum_{\sum_{i=1}^{n-1} S_i = k^{(n-1)}} \prod_{i=1}^{n-1} F_i(S_i). F_n(S_n)$$

$G_n\left(k^{(n)}\right) = \sum_{k^{(n-1)} + S_n = k^{(n)}} G_{n-1}\left(k^{(n-1)}\right). F_n(S_n)$ (7)

Equation (7), together with the initial conditions $G_1(.) = F_1(.)$. completes the description of the algorithm [12] and [20]. For the normalization constant we have

$G(K) = G_N(K^{(N)})$ (8)

According to the marginal probability that there are exactly $S_i = k$ jobs at node i is given by

$$p_i(k) = \sum_{\substack{\sum_{i=1,}^{N} S_j = K \\ S_i = k}} p(S_1, S_2 \ldots, S_N) = \frac{1}{G(K)} \sum_{\substack{\sum_{j=1,}^{N} S_j = K \\ S_j = k}} \prod_{j=1}^{N} F_j(S_j)$$

$= \frac{F_i(k)}{G(K)} \sum_{\substack{\sum_{j=1,}^{N} S_j = K \\ S_j = k}} \prod_{\substack{j=1, \\ j \neq i}}^{N} F_j(S_j) = \frac{F_i(k)}{G(K)} G_N^{(i)}(K - k)$ (9)

Then $G_N^{(i)}(K)$ can again be interpreted as the normalization constant of the network without node i:

<table>
<tr>
<td>Step 2.3: For i =1, ….,N and r =1,….. R, compute the mean number of class-r jobs at the ith node:

$K_{ir}(k) = \lambda_r(k).T_{ir}(k).e_{ir}$</td>
<td>$G_N^{(i)}(k) = \sum_{\sum_{j=1, S_j=K-k}^{N} S_j = K} \prod_{j=1, j\neq i}^{N} F_j(S_j)$ (10)

The iterative formula for computing the normalization constant $G_N^{(i)}(k)$ is given as follows:
$G_N^{(i)}(k) = G(k) - \sum_{j=0}^{k} \delta(j).F_i(j).G_N^{(i)}(k-j).$ (11)
With δ(j) defined by

$\delta(j) = \begin{cases} 0, & if \quad j = 0, \\ 1, & otherwise \end{cases}$ (12)
and affecting the computation only j = 0. The initial condition is

$G_N^{(i)}(0) = G(0) = 1,$ i = 1, 2, …….., N.</td>
</tr>
</table>

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